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1976 J. Phys. A: Math. Gen. 9 1887

(http://iopscience.iop.org/0305-4470/9/11/013)

J. Phys. A: Math. Gen., Vol. 9, No. 11, 1976. Printed in Great Britain. © 1976

A class of algebraically general non-null Einstein–Maxwell fields

Alan Barnes

School of Mathematics, 39 Trinity College, Dublin University, Dublin 2, Ireland
and
School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

Received 21 June 1976

Abstract. A class of solutions of the Einstein–Maxwell field equations which satisfy the condition that the null tetrad, determined by the electromagnetic field, is parallelly transported along the curves tangent to the complex null vectors m_i and $\bar{m}_i$ is investigated. For the case when m_i has zero twist the general solution of the field equations is found in closed form. A symmetry of the Newman–Penrose equations first introduced by Sachs is used to show the relationship between this class of solutions and those recently studied by Tariq and Tupper.

1. Introduction

A class of solutions of the vacuum Einstein–Maxwell field equations is investigated. The electromagnetic field will be assumed to be non-null and consequently a pseudo-orthonormal tetrad of two real null vectors l_i and n_i and two complex-conjugate vectors m_i and $\bar{m}_i$ exists such that the self-dual Maxwell bivector F^+_{ij} takes the form:

$$F^+_{ij} = \phi(n_{[i}l_{j]} + m_{[i}\bar{m}_{j]}), \tag{1.1}$$

where ϕ is the complex electromagnetic field strength. Maxwell's equations can be written in the concise form:

$$F^{+ij}{}_{;j} = 0. \tag{1.2}$$

The fields investigated here are assumed to satisfy the condition that all four vectors of the null tetrad are parallelly transported along the curves tangent to m_i and $\bar{m}_i$, i.e.

$$l_{i;j}m^j = n_{i;j}m^j = m_{i;j}m^j = \bar{m}_{i;j}m^j = 0. \tag{1.3}$$

These fields are shown to be algebraically general in Petrov's classification and the spin coefficients τ and π (the complex expansions of m_i and $\bar{m}_i$) are coupled by the relation $\tau\bar{\tau} = \pi\bar{\pi}$. No solutions exist for which m_i is expansion-free but twisting ($\tau = \bar{\pi}$) whereas in the case where m_i is twist-free but expanding ($\tau + \bar{\pi} = 0$) the general solution can be obtained in closed form.

Einstein–Maxwell fields satisfying the dual condition that the null tetrad is parallelly propagated along l_i and n_i have many analogous properties and have recently been investigated by Tariq and Tupper (1974, 1975, to be referred to as TT). For example, their solutions are also algebraically general and the complex expansions ρ and μ of l_i and n_i satisfy the coupling theorem of Debney and Zund (1972) namely $\rho\bar{\mu} = \bar{\rho}\mu$.

The two classes of solution are related by a duality transformation first introduced by Sachs. The existence of this duality relationship between general classes of non-null Einstein–Maxwell fields and its interpretation as a tetrad transformation in a complex space–time are briefly discussed.

2. The basic equations and the duality transformation

Throughout this paper the spin coefficient formalism and notation of Newman and Penrose (1962, 1963) will be used.

On choosing the null tetrad to be aligned with the electromagnetic field as in equation (1.1) we obtain $\phi_0 = \phi_2 = 0$; $\phi_1 = \phi$. The conditions that the tetrad is parallelly propagated along m_i are

$$\rho = \sigma = \mu = \lambda = \alpha = \beta = 0. \tag{2.1}$$

Maxwell's equations (1.2) take the particularly simple form:

$$\mathrm{D}\phi = \Delta\phi = (\delta - 2\tau)\phi = (\bar{\delta} + 2\pi)\phi = 0. \tag{2.2}$$

The Ricci identities become:

$$\delta\tau = \tau^2 - \kappa\bar{\nu} \tag{2.3a}$$

$$\delta\kappa = (\tau - \bar{\pi})\kappa - \psi_0 \tag{2.3b}$$

$$\psi_1 = 0 \tag{2.3c}$$

$$\delta\epsilon = \gamma\kappa - \bar{\pi}\epsilon \tag{2.3d}$$

$$\delta\gamma = \tau\gamma - \epsilon\bar{\nu} \tag{2.3e}$$

$$\psi_2 - \phi\bar{\phi} = 0 \tag{2.3f}$$

$$\delta\nu = \nu\tau - \bar{\nu}\pi \tag{2.3g}$$

$$\delta\pi = \nu\kappa - \pi\bar{\pi} - \psi_2 \tag{2.3h}$$

$$\psi_3 = 0 \tag{2.3i}$$

$$\bar{\delta}\tau = \tau\bar{\tau} - \nu\kappa + \psi_2 \tag{2.3j}$$

$$\bar{\delta}\kappa = \bar{\kappa}\tau - \kappa\pi \tag{2.3k}$$

$$\bar{\delta}\epsilon = \bar{\kappa}\gamma - \epsilon\pi \tag{2.3l}$$

$$\bar{\delta}\gamma = \bar{\tau}\gamma - \epsilon\nu \tag{2.3m}$$

$$\bar{\delta}\nu = (\bar{\tau} - \pi)\nu + \psi_4 \tag{2.3n}$$

$$\bar{\delta}\pi = \bar{\kappa}\nu - \pi^2 \tag{2.3p}$$

$$\mathrm{D}\tau - \Delta\kappa = (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma})\kappa \tag{2.3q}$$

$$\mathrm{D}\nu - \Delta\pi = (\gamma - \bar{\gamma})\pi - (3\epsilon + \bar{\epsilon})\nu \tag{2.3r}$$

$$\mathrm{D}\gamma - \Delta\epsilon = -(\epsilon + \bar{\epsilon})\gamma - (\gamma + \bar{\gamma})\epsilon + \tau\pi - \nu\kappa + \psi_2 + \phi\bar{\phi}. \tag{2.3s}$$

On noting equations (2.1), (2.2) and (2.3c,f,i) the Bianchi identities reduce to the form:

$$\Delta\psi_0 = 4\gamma\psi_0 \tag{2.4a}$$

$$\delta\psi_0 = -\pi\psi_0 + \kappa\psi_2 \tag{2.4b}$$

$$\nu\psi_0 = (2\bar{\pi} - \tau)\psi_2 \tag{2.4c}$$

$$\kappa\psi_4 = (2\bar{\tau} - \pi)\psi_2 \tag{2.4d}$$

$$\mathrm{D}\psi_2 = \Delta\psi_2 = 0 \tag{2.4e}$$

$$\delta\psi_4 = \tau\psi_4 - \nu\psi_2 \tag{2.4f}$$

$$\mathrm{D}\psi_4 = -4\epsilon\psi_4. \tag{2.4g}$$

The intrinsic differential operators satisfy the following commutation relations:

$$\bar{\delta}\delta - \delta\bar{\delta} = 0 \tag{2.5a}$$

$$\delta\mathrm{D} - \mathrm{D}\delta = -\bar{\pi}\mathrm{D} + \kappa\Delta - (\epsilon - \bar{\epsilon})\delta \tag{2.5b}$$

$$\delta\Delta - \Delta\delta = -\bar{\nu}\mathrm{D} + \tau\Delta - (\gamma - \bar{\gamma})\delta \tag{2.5c}$$

$$\Delta\mathrm{D} - \mathrm{D}\Delta = (\gamma + \bar{\gamma})\mathrm{D} + (\epsilon + \bar{\epsilon})\Delta - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta. \tag{2.5d}$$

In fact the above equations can be obtained directly from equations (2.4), (2.5), (2.6), (2.9), (3.1) and (3.2) of TT by means of a duality transformation. This operation is induced by the quasi-tetrad transformation:

$$l_i^* = m_i, \qquad n_i^* = -\bar{m}_i, \qquad m_i^* = l_i, \qquad \bar{m}_i^* = -n_i.$$

The operation preserves the tetrad orthogonality relations (e.g. $l_i^* n^{i*} = -m_i\bar{m}^i = +1$) and a double application of the operation is the identity. In general the operation does not commute with complex conjugation. The spin coefficients transform under (*) as follows:

$$\begin{array}{llll} \kappa^* = -\sigma, & \bar{\kappa}^* = -\bar{\lambda}, & \rho^* = \tau, & \bar{\rho}^* = -\bar{\pi}, \\ \nu^* = \lambda, & \bar{\nu}^* = \bar{\sigma}, & \mu^* = -\pi, & \bar{\mu}^* = \bar{\tau}, \\ \alpha^* = -\gamma, & \bar{\alpha}^* = -\bar{\epsilon}, & \beta^* = \epsilon, & \bar{\beta}^* = \gamma. \end{array} \tag{2.6}$$

The transformation properties of the remaining spin coefficients can be obtained by operating on equation (2.6) with the duality operator (*) and using the fact that for any spin coefficient x, $x^{**} = x$.

The behaviour of the components of the Maxwell and Weyl tensors and of the intrinsic derivative operators is given by

$$\begin{array}{llll} \phi_0^* = -\phi_0, & \bar{\phi}_0^* = -\bar{\phi}_2, & \phi_2^* = -\phi_2, & \bar{\phi}_2^* = -\phi_0, \\ \phi_1^* = \phi_1, & \bar{\phi}_1^* = -\bar{\phi}_1, & \psi_2^* = \psi_2, & \bar{\psi}_2^* = \bar{\psi}_2, \\ \psi_0^* = \psi_0, & \bar{\psi}_0^* = \bar{\psi}_4, & \psi_1^* = -\psi_1, & \bar{\psi}_1^* = -\bar{\psi}_3, \\ \psi_4^* = \psi_4, & \bar{\psi}_4^* = \bar{\psi}_0, & \psi_3^* = -\psi_3, & \bar{\psi}_3^* = -\bar{\psi}_1, \\ \mathrm{D}^* = \delta, & \Delta^* = -\bar{\delta}, & \delta^* = \mathrm{D}, & \bar{\delta}^* = -\Delta. \end{array} \tag{2.7}$$

The whole spin coefficient formalism is invariant under the (*) operation. For example the Ricci identity involving $\mathrm{D}\rho - \bar{\delta}\kappa$ is transformed to that involving $\delta\tau - \Delta\sigma$ whereas the identity containing $D\sigma - \delta\kappa$ is mapped to itself. The duality operator preserves the class of vacuum fields and also the class of non-null (but not null) Einstein–Maxwell fields. As the spin coefficient formalism is invariant under the operation it is often possible to write down dual versions of theorems (and their proofs) relating to vacuum or non-null Einstein–Maxwell solutions.

In particular the solutions considered by TT which satisfy the condition that $\tau = \kappa = \nu = \pi = \epsilon = \gamma = 0$ are dual to those considered in this paper. Dual versions of the TT theorems will be given in § 3 without explicit proofs.

As a further example we note that the conditions $\kappa = \nu = 0$ and $\psi_0 = \psi_1 = 0$ are both self-dual. Hence the vacuum Goldberg–Sachs theorem is self-dual and furthermore so is its proof (Newman and Penrose 1962).

However one point should not be overlooked when constructing dual proofs: a proof which relies on the reality of some quantity $x\bar{x}$ or l^i will not in general be valid in dualized form, as x^* and $\bar{x}^*$ will not usually be complex conjugates. This is due to the fact that the (*) operation is not a valid tetrad transformation for real space–times. However if one considers complex space–times with a basis of four independent complex null vectors l_i, n_i, m_i and $\bar{m}_i$ then the operation is a valid tetrad transformation. In this case barred and unbarred quantities are no longer restricted to be complex conjugates and the Einstein–Maxwell field equations (and their complex conjugates) are replaced by a set of formally identical equations for 24 complex spin coefficients (Fette *et al* 1976). The (*) operation, being a tetrad transformation, leaves invariant these complex equations.

3. General results

By applying the commutation relations (2.5) to the scalar ϕ and noting equation (2.2) we obtain the equations:

$$\tau\bar{\tau} = \pi\bar{\pi} \tag{3.1a}$$

$$D\tau = (\epsilon - \bar{\epsilon})\tau \tag{3.1b}$$

$$D\pi = (\bar{\epsilon} - \epsilon)\pi \tag{3.1c}$$

$$\Delta\tau = (\gamma - \bar{\gamma})\tau \tag{3.1d}$$

$$\Delta\pi = (\bar{\gamma} - \gamma)\pi \tag{3.1e}$$

$$D\nu = -(3\epsilon + \bar{\epsilon})\nu \tag{3.1f}$$

$$\Delta\kappa = (3\gamma + \bar{\gamma})\kappa. \tag{3.1g}$$

On differentiating the Bianchi identities (2.4*c*,*d*) we obtain:

$$D\kappa = (3\epsilon + \bar{\epsilon})\kappa \tag{3.2a}$$

$$\Delta\nu = -(3\gamma + \bar{\gamma})\nu \tag{3.2b}$$

$$\Delta\psi_4 = -4\gamma\psi_4 \tag{3.2c}$$

$$D\psi_0 = 4\epsilon\psi_0 \tag{3.2d}$$

$$\psi_0\psi_4 = -\psi_2(4\tau\bar{\tau} - 2\bar{\tau}\bar{\pi} - 2\bar{\kappa}\bar{\nu} + 3\psi_2). \tag{3.2e}$$

Equations (3.1) and (3.2) above are completely analogous to equations (2.7), (2.8) and

(3.7) of TT. In particular equation (3.1a) is the analogue of the coupling theorem of Debney and Zund (1972) and states that the complex expansions τ and π of m_i and $\bar{m}_i$ have the same magnitude. This result can be proved under the weaker assumptions $\rho+\bar{\rho}=\mu+\bar{\mu}=0$.

Theorem 1. There are no fields for which m_i is expansion-free but twisting.

Proof. The condition that m_i is expansion-free but twisting is $\tau=\bar{\pi}\neq 0$. From equations (2.3a,h,j,p) we deduce that $\tau^2=\kappa\bar{\nu}$ and $2\psi_2=\nu\kappa+\bar{\nu}\bar{\kappa}-2\tau\bar{\tau}$. Hence $|\kappa\nu|=\tau\bar{\tau}$ and so $\psi_2\leqslant 0$. However equation (2.3f) implies $\psi_2\geqslant 0$ and consequently $\psi_2=\phi=0$. This result has no analogue in TT as it depends on the inequality $\tau\bar{\tau}\geqslant 0$ which has no dual version. We now list three results which are completely dual to those in TT.

Lemma 1. The vanishing of any one of the spin coefficients τ, π, κ and ν implies that all four vanish and furthermore that the electromagnetic field is zero.

Theorem 2. A non-null Maxwell field which is such that the null tetrad determined by the electromagnetic field is parallelly transported along m_i is algebraically general.

In fact by a slight extension of the argument in TT and its dual one can show that if $\alpha\psi_2^2=\psi_0\psi_4$ where α is a real constant, then $\alpha=-3$ ($\alpha=9$ would give an algebraically special field).

Theorem 3. A necessary and sufficient condition that one can put $\epsilon=\gamma=0$ by means of an allowable tetrad transformation: $\tilde{l}_i=Al_i$, $\tilde{n}_i=A^{-1}n_i$, $\tilde{m}_i=e^{i\theta}m_i$ with $\delta A=\delta\theta=0$; is that m_i is non-twisting (i.e. $\tau+\bar{\pi}=0$).

4. An exact solution

We assume that $\tau+\bar{\pi}=0$ and hence by theorem 3 we may put $\epsilon=\gamma=0$. Consequently equation (2.3s) implies that $2\psi_2=\nu\kappa+\tau\bar{\tau}$ and $\nu\kappa=\bar{\nu}\bar{\kappa}$. From equations (2.3$c,d$) and (3.2$e$) we deduce that

$$\psi_2=2\tau\bar{\tau}=\tfrac{2}{3}\nu\kappa \tag{4.1a}$$

$$\psi_0=-2\tau\kappa \tag{4.1b}$$

$$\psi_4=2\bar{\tau}\nu. \tag{4.1c}$$

Applying the differential operator δ to (4.1a) one obtains

$$\tau^2=-3\kappa\bar{\nu}. \tag{4.2}$$

Hence we see that (2.3), (2.4), (2.5), (3.1) and (3.2) imply that the intrinsic derivatives D and Δ of κ, ν, τ, π, ψ_0, ψ_2, and ψ_4 are all zero and that

$$\delta\tau=4\tau^2 \tag{4.3a}$$

$$\delta\kappa=4\tau\kappa \tag{4.3b}$$

$$\bar{\delta}\nu=4\bar{\tau}\nu \tag{4.3c}$$

$$\bar{\delta}\kappa=\bar{\delta}\tau=\delta\nu=0. \tag{4.3d}$$

The vectors m_i and $\bar{m}_i$ have vanishing Lie bracket as each is parallelly transported along the other. Furthermore (3.1) together with the conditions $\tau+\bar{\pi}=\epsilon=\gamma=0$ imply that $m_{[i;j]}=0$ (i.e. m_i is a gradient). Consequently a complex coordinate $\xi=x^3+\mathrm{i}x^4$ exists such that $m_i=\xi_{,i}$ and $m^i(\partial/\partial x^i)=(\partial/\partial\xi)$. From the orthogonality relations we deduce that $l^3=l^4=n^3=n^4=0$. The commutation relations (2.5) applied to the coordinates $x^\alpha(\alpha=1,2)$, lead to

$$\delta l^\alpha=\tau l^\alpha+\kappa n^\alpha \tag{4.4a}$$

$$\delta n^\alpha=-\bar{\nu} l^\alpha+\tau n^\alpha \tag{4.4b}$$

$$\mathrm{D}n^\alpha=\Delta l^\alpha. \tag{4.4c}$$

Equations (4.1)–(4.4) are sufficient to completely determine the metric and tetrad vectors. Integration of these equations is relatively straightforward and after using the allowable coordinate and tetrad freedom to simplify the results we obtain

$$\begin{gathered}\tau=-\frac{1}{4\xi},\qquad \kappa=\frac{\mathrm{i}\sqrt{3}}{4\xi},\qquad \nu=-\frac{\mathrm{i}\sqrt{3}}{4\bar{\xi}}\\ \psi_0=\frac{\mathrm{i}\sqrt{3}}{8\xi^2},\qquad \psi_2=\frac{1}{8\xi\bar{\xi}},\qquad \psi_4=\frac{\mathrm{i}\sqrt{3}}{8\bar{\xi}^2}\\ l^i=r^{-1/2}\left(\cos\frac{\sqrt{3}}{2}\theta,\ -\sin\frac{\sqrt{3}}{2}\theta,0,0\right)\\ n^i=r^{-1/2}\left(\sin\frac{\sqrt{3}}{2}\theta,\cos\frac{\sqrt{3}}{2}\theta,0,0\right)\end{gathered} \tag{4.5}$$

where we have introduced real coordinates r and θ by means of the relation $\xi=r\mathrm{e}^{\mathrm{i}\theta}$. With the aid of the completeness relation $g^{ij}=2l^{(i}n^{j)}-2m^{(i}\bar{m}^{j)}$ it is now a simple matter to determine the metric:

$$\mathrm{d}s^2=r[\sin\sqrt{3}\theta(\mathrm{d}x^1)^2-2\cos\sqrt{3}\theta\,\mathrm{d}x^1\,\mathrm{d}x^2-\sin\sqrt{3}\theta(\mathrm{d}x^2)^2]-\mathrm{d}r^2-r^2\,\mathrm{d}\theta^2. \tag{4.6}$$

The electromagnetic field is determined by integrating Maxwell's equations (2.2). The result is

$$\phi=\mathrm{e}^{\mathrm{i}p}/2\sqrt{2}r \tag{4.7}$$

where p is an arbitrary constant determining the complexion of the electromagnetic field.

The exact solution represented by (4.5)–(4.7) is very special and contains no arbitrary parameters apart from the trivial constant p which does not affect the gravitational field. However it is algebraically general in terms of the Petrov classification as $3\psi_2^2=-\psi_0\psi_4\neq 0$. The solution possesses a real curvature singularity at $r=0$ as both the invariants $\phi\bar{\phi}$ and ψ_2 become infinite there. The solution admits a three-parameter isometry group with three-dimensional orbits generated by the Killing vectors

$$X_1=\frac{\partial}{\partial x^1},\qquad X_2=\frac{\partial}{\partial x^2},\qquad X_3=\frac{\sqrt{3}}{2}\left(x^2\frac{\partial}{\partial x^1}-x^1\frac{\partial}{\partial x^2}\right)+\frac{\partial}{\partial\theta} \tag{4.8}$$

with commutation relations of an algebra of Bianchi type VII:

$$[X_1,X_2]=0,\qquad [X_1,X_3]=-\frac{\sqrt{3}}{2}X_2,\qquad [X_2,X_3]=\frac{\sqrt{3}}{2}X_1. \tag{4.9}$$

The isometry group is complete as $\psi_2 = (1/8r^2)$ is a metric invariant and a direct integration of Killing's equation reveals that X_1, X_2 and X_3 are the only Killing vectors tangent to the hypersurfaces $r = \text{constant}$.

The causal nature of all of the Killing vectors is not the same everywhere. For example $\partial/\partial x^1$ is time-like, null or space-like when $\sin\sqrt{3}\theta$ is positive, zero or negative respectively. The hypersurfaces where it is null (i.e. $\sin\sqrt{3}\theta = 0$) are time-like and so are not Killing horizons in the sense of Carter (1973). None of the Killing vectors is hypersurface orthogonal and so the solution is not static but only locally stationary.

The range of the coordinates is as follows:

$$-\infty < x^1 < +\infty, \qquad -\infty < x^2 < +\infty, \qquad 0 < r < +\infty, \qquad -\infty < \theta < +\infty.$$

The coordinate r could take negative values but in this case the resultant space–time would not be connected and would consist of two identical components $r < 0$ and $r > 0$. One could of course make any of the coordinates x^1, x^2, θ periodic by considering a quotient manifold of the original space–time manifold by a discrete subgroup of the isometry group. Provided that the action of this subgroup is free and properly discontinuous then the quotient manifold will be a Hausdorff space–time (Hawking and Ellis 1973). For example one could make θ periodic with period s_0 by identifying all points of the form

$$\begin{aligned}
\tilde{x}^1 &= \cos\left(\frac{\sqrt{3}}{2}s\right)x^1 + \sin\left(\frac{\sqrt{3}}{2}s\right)x^2 \\
\tilde{x}^2 &= -\sin\left(\frac{\sqrt{3}}{2}s\right)x^1 + \cos\left(\frac{\sqrt{3}}{2}s\right)x^2 \\
\tilde{r} &= r \\
\tilde{\theta} &= \theta + s
\end{aligned} \tag{4.10}$$

where s is an arbitrary integral multiple of s_0. The 2-surfaces $x^1 = \text{constant}$, $x^2 = \text{constant}$ are then intrinsically flat and are globally isometric to a (possibly many-sheeted) cone with the vertex removed, or if $s_0 = 2\pi$ to a punctured plane. Unless s_0 is an integral multiple of $4\pi/\sqrt{3}$ the Killing vector fields X_1 and X_2 are not invariant under the subgroup (4.10) of the one-parameter group generated by X_3. Consequently X_1 and X_2 in general do not generate global isometries in the quotient space–time though they are still of course local Killing vectors.

The space–time is not asymptotically flat and so it is not possible to interpret the metric as representing the field of a spinning particle. As yet no physical interpretation for the metric is known.

Acknowledgments

The author would like to thank P S Florides and A MacDonald for helpful discussions and the Department of Education, Dublin for generous financial support.

Note added in proof. The metric (4.6) cannot be written in the usual Lewis–Papapetrou form for stationary axisymmetric fields:

$$ds^2 = f(\rho, z)(dt + \omega(\rho, z)\, d\phi)^2 - f^{-1}(\rho, z)[\rho^2\, d\phi^2 + e^{2\gamma(\rho,z)}(d\rho^2 + dz^2)],$$

because the electromagnetic field circularity conditions (Carter 1973) are not satisfied and consequently

$$R^1_1 + R^2_2 \neq 0.$$

Although the metric and Ricci tensor satisfy their respective circularity conditions, the generalized Papapetrou theorem (Carter 1973) is not violated as the axis ($r = 0$) is singular.

References

Carter B 1973 *Les Astres Occlus* eds B S De Witt and C De Witt (New York: Gordon and Breach)
Debney G C and Zund J 1972 *Tensor New Ser.* **25** 53
Fette C W, Janis A I and Newman E T 1976 *J. Math. Phys.* **17** 660
Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Space–Time* (Cambridge: Cambridge University Press) p 173
Newman E T and Penrose R 1962 *J. Math. Phys.* **3** 566
—— 1963 *J. Math. Phys.* **4** 998
Tariq N and Tupper B O J 1974 *Tensor New Ser.* **28** 83
—— 1975 *Gen. Rel. Grav.* **6** 345